

DIMENSION FUNCTIONS ON THE SPECTRUM OVER BOUNDED GEODESICS AND APPLICATIONS TO DIOPHANTINE APPROXIMATION

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ABSTRACT. The set of geodesic rays avoiding a suitable obstacle in a complete negatively curved Riemannian manifold determines a spectrum \mathcal{S} . While various properties of this spectrum are known, we define and study dimension functions on \mathcal{S} in terms of the Hausdorff-dimension of suitable subsets of the set of bounded geodesic rays. We establish estimates on the Hausdorff-dimension of these subsets and thereby obtain non-trivial bounds for the dimension functions. Finally, we apply the obtained results to the dimension functions on the spectrum of complex numbers badly approximable by either an imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$ or by quadratic irrational numbers over $\mathbb{Q}(i\sqrt{d})$.

1. INTRODUCTION AND RESULTS

1.1. Outline. Let M be a complete connected Riemannian manifold of curvature at most -1 . The investigation of geodesics in M avoiding an obstacle has been studied in various contexts and is often deeply connected to problems in Diophantine approximation; see for instance [15, 7] and references therein. Following the geometric viewpoint developed in these works (as well as in earlier ones such as [8, 17, 20, 21, 22]) we continue the investigation as follows which will be made precise in the respective subsections below:

The geodesic flow $\phi^t : SM \times \mathbb{R} \rightarrow SM$ acts on the unit tangent bundle SM of M . For a vector $v \in SM$ we call the orbit $\gamma_v(t) \equiv \pi \circ \phi^t(v)$, $t \geq 0$, a geodesic ray in M (where $\pi : SM \rightarrow M$ denotes the footpoint projection). Given an obstacle O (such as a 'cusp', a point, or a closed geodesic) call a geodesic ray in M *bounded* if it avoids a suitable neighborhood of the obstacle (given in terms of a height, distance or length functional). To each bounded geodesic ray γ_v we assign a real constant $c(v)$, the *approximation constant* (defined by the respective functional), and the set of bounded geodesic rays starting in a point (or another set) determines the *spectrum* $\mathcal{S} \subset \mathbb{R}$.

Considering the modular surface $M = \mathbb{H}^2/PSL(2, \mathbb{Z})$ and letting O be the cusp of M , \mathcal{S} is related to the classical *Markoff spectrum* \mathcal{M} . Recall from [4] that the *Lagrange spectrum* $\mathcal{L} \equiv \{c^+(x)^{-1} : x \in \mathbf{Bad}\} \subset \mathbb{R}$ and the Markoff spectrum $\mathcal{M} \equiv \{c(x)^{-1} : x \in \mathbf{Bad}\}$ are determined by the approximation constants

$$c^+(x) \equiv \liminf_{p,q \rightarrow \infty} q^2 |x - \frac{p}{q}|, \quad c(x) \equiv \inf_{(p,q) \in \mathbb{Z} \times \mathbb{N}} q^2 |x - \frac{p}{q}| \quad (1.1)$$

for *badly approximable* numbers x in $\mathbf{Bad} = \{x \in \mathbb{R} : c(x) > 0\}$. The spectrum \mathcal{L} is

1. bounded below by the *Hurwitz constant* $\mathfrak{h} \equiv \inf \mathcal{L} = \sqrt{5}$ (Hurwitz 1875),
2. contains a *Hall ray* (Hall 1947, see below),
3. equals the closure of the set $\{c^+(x)^{-1} : x \in \mathcal{P}\}$ where \mathcal{P} denotes the quadratic irrational numbers over \mathbb{Q} (Cusick 1975); in particular, \mathcal{L} is closed and

$$\frac{1}{\mathfrak{h}} = \sup_{x \in \mathcal{P}} c(x).$$

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Moreover, the Lagrange spectrum is a subset of the Markoff spectrum where the inclusion is proper and the intersection $\mathcal{L} \cap \mathcal{M}$ contains a positive half-line.

While various of the above properties were established also for the spectra of interest in our paper, see for instance [7, 10, 13, 14, 16, 22], the main intention of this paper is the following. Define the *dimension functions* $\mathfrak{D}, \mathfrak{D}^0 : \mathcal{S} \rightarrow \mathbb{R}$ on the spectrum \mathcal{S} via the Hausdorff-dimension of the sublevelset, respectively the levelset, of the assignment c . We study these dimension functions for which we establish nontrivial bounds; see Section 1.2. For this we consider suitable subsets of the set of bounded geodesic rays and estimate their Hausdorff-dimension.

Finally, exploiting the connection between the dynamics of geodesic rays in Bianchi orbifolds to Diophantine approximation of complex numbers badly approximable by imaginary quadratic number fields or by quadratic irrational numbers over such, we obtain nontrivial bounds for the dimension functions on the corresponding spectra; see Section 1.3. We conclude Section 1 by further discussion and, in order to keep the exposition readable, skip all the proofs of Section 1.2 to Section 2.

1.2. Bounded geodesic rays in negatively curved manifolds. Many of the following setups can be considered in a more general context, for instance when M is geometrically finite, pinched or negatively curved, or even is a quotient of a proper geodesic $\text{CAT}(-1)$ metric space. However, unless stated otherwise, we assume for simplicity that M is a complete $(n+1)$ -dimensional hyperbolic finite volume Riemannian manifold. As a general reference for the following see [23] as well as Section 2.1.

1.2.1. Avoiding a cusp. Let M be noncompact and let e be a *cusp* of M , that is to say an asymptotic class of minimizing geodesic rays along which the injectivity radius tends to 0. Let β_e be a Busemann function on M (associated to such a minimizing geodesic ray) such that $H_t \equiv \beta_e^{-1}((t, \infty))$ gives shrinking neighborhoods of the cusp as $t \rightarrow \infty$, which serves as a *height function*. Up to renormalizing β_e assume that H_0 is a sufficiently small cusp neighborhood (see Section 2.1 for definitions).

Let SH_0^+ denote the n -dimensional submanifold of SM consisting of outward unit vectors orthogonal to ∂H_0 . Each vector in $v \in SH_0^+$ can be identified with a geodesic line in M starting from the cusp with $\gamma_v(0) \in \partial H_0$. Define for a vector $v \in SH_0^+$ the *height constant* to the cusp e by

$$\mathcal{H}(v) \equiv \sup_{t \geq 0} \beta_e(\gamma_v(t)) \in \mathbb{R} \cup \{\infty\}. \quad (1.2)$$

Note that for a typical $v \in SH_0^+$ we have that γ_v is unbounded with $\mathcal{H}(v) = \infty$.¹ Conversely a vector $v \in SH_0^+$ is called *bounded* (with respect to the cusp e) if $\mathcal{H}(v) < \infty$. By [12], the set of bounded vectors $v \in SH_0^+$ is of Hausdorff-dimension n and in fact an *absolute winning set* (see below); see [24] for further generalizations. Define the *height spectrum* of the data (M, e, β_e) by $\mathcal{S}_{\mathcal{H}} \equiv \{\mathcal{H}(v) : v \in SH_0^+ \text{ bounded}\}$. We define two *dimension height functions* $\mathfrak{D}_{\mathcal{H}}, \mathfrak{D}_{\mathcal{H}}^0 : \mathcal{S}_{\mathcal{H}} \rightarrow [0, n]$ on the spectrum $\mathcal{S}_{\mathcal{H}}$ by

$$\begin{aligned} \mathfrak{D}_{\mathcal{H}}(t) &\equiv \dim(\{v \in SH_0^+ : \mathcal{H}(v) \leq t\}), \\ \mathfrak{D}_{\mathcal{H}}^0(t) &\equiv \dim(\{v \in SH_0^+ : \mathcal{H}(v) = t\}), \end{aligned} \quad (1.3)$$

¹ When M has only one cusp, this follows for instance from Sullivan's *logarithm law* [20]: for almost all (spherical measure) vectors $v \in SM_o$, where o in M is a base point, we have $\limsup_{t \rightarrow \infty} \frac{d(\gamma_v(t), o)}{\log(t)} = \frac{1}{n}$.

where 'dim' stands (here and hereafter) for the Hausdorff-dimension. Clearly, for all $t \in \mathcal{S}_{\mathcal{H}}$,

$$0 \leq \mathfrak{D}_{\mathcal{H}}^0(t) \leq \mathfrak{D}_{\mathcal{H}}(t) \leq n.$$

Thus, if $t \in \mathcal{S}_{\mathcal{H}}$ is a given height constant then $\mathfrak{D}_{\mathcal{H}}(t)$ equals the Hausdorff-dimension of the set

$$\mathcal{B}_{M,e,\beta_e}(t) \equiv \{v \in SH_0^+ : \gamma_v(s) \notin H_t \text{ for all } s \geq 0\},$$

corresponding to the set of rays γ_v avoiding the cusp neighborhood H_t of e .

Remark. Consider the asymptotic height spectrum $\mathcal{S}_{\mathcal{H}}^+$ of M (geometrically finite, negatively curved) instead, that is the spectrum of asymptotic height constants $\mathcal{H}^+(v)$, where we use the 'limsup' in (1.2), and restrict to positively recurrent vectors in SH_0^+ ; a vector $v \in SM$ is *positively recurrent* if the ray γ_v hits a compact set K in M infinitely many times. Then the Properties 1. - 3. as above hold, where we replace \mathcal{P} by the set of periodic vectors in SM , and the Hurwitz constant can be determined explicitly in some concrete examples; see [10, 14, 15, 22] respectively. Note that $\mathcal{H}^+(v) \leq \mathcal{H}(v)$ for every $v \in SH_0^+$. Hence, defining the *asymptotic dimension height function* $\mathfrak{D}_{\mathcal{H}^+}$ in a similar way to (1.3) with respect to \mathcal{H}^+ , we obtain $\mathfrak{D}_{\mathcal{H}^+}(t) \geq \mathfrak{D}_{\mathcal{H}}(t)$ for all $t \in \mathcal{S}_{\mathcal{H}} \cap \mathcal{S}_{\mathcal{H}}^+$.

From the author's earlier work [26], when M has only one cusp, there exist a height t_0 and constants $k_u, k_l > 0$ such that for all $t_0 \leq t \in \mathcal{S}_{\mathcal{D}}$ we have

$$n - \frac{k_l}{t \cdot e^{n/2t}} \leq \mathfrak{D}_{\mathcal{H}}(t) \leq n - \frac{k_u}{t \cdot e^{2nt}}. \quad (1.4)$$

Remark. In light of the correspondence between badly approximable real numbers and bounded geodesic rays in the modular surface $M = \mathbb{H}^2/PSL(2, \mathbb{Z})$, (1.4) generalizes a classical inequality of Jarník [9] and is called a *Jarník-type inequality* by analogy in [26]. A similar inequality holds when M is geometrically finite, restricting to positively recurrent vectors.

We next establish non-trivial bounds for $\mathfrak{D}_{\mathcal{H}}^0(t)$. When M has precisely one cusp, each bounded geodesic ray γ_v , $v \in SH_0^+$, determines a countable discrete set of times $\{t_i(v) : i \in \mathbb{N}\} \subset [0, \infty)$ of local maxima of the height function β_e with corresponding heights $h_i(v) = \beta_e(\gamma_v(t_i(v)))$. If M has more cusps, then possibly a subray of γ_v may diverge to another cusp and we simply set $t_i(v) = -\infty$ for sufficiently large i . Given parameters $c_0 \geq 0$ and $s_0 \geq 0$, define the set of bounded vectors for which the first height h_1 equals c_0 and all others are bounded by s_0 ,

$$S(c_0, s_0) \equiv \{v \in SH_0^+ : h_1(v) = c_0, \ h_i(v) \leq s_0 \text{ for all } i \in \mathbb{N}_{\geq 2}\}$$

Schmidt, Sheingorn [18] showed for $n = 1$ and Parkkonen, Paulin [15] for $n \geq 2$ and curvature at most -1 that $S(c_0, \bar{s}_0)$ is nonempty for all sufficiently large lengths $c_0 \geq \bar{c}_0$ and some constant \bar{s}_0 . When $S(c_0, c_0)$ is nonempty for all sufficiently large $c_0 \geq t_0$, this implies the existence of a *Hall ray* at the cusp, that is, there exists a height $t_0 \in \mathbb{R}$ such that $[t_0, \infty) \subset \mathcal{S}_{\mathcal{H}}$.²

Our first theorem establishes a lower bound on the dimensions of the sets $S(c_0, s_0)$.

² Note that a bound on the height t_0 was determined explicitly with $t_0 = 4.16$ for $n = 1$ in [18], and with $t_0 = 4.2$ for $n \geq 2$ in [15].

Theorem 1.1. *Let $n \geq 2$. There exists a height $t_0 \geq 0$ and a positive constant $k_0 > 0$, both independent of s_0 and c_0 , such that for all heights c_0 and heights $s_0 \geq t_0$, the Hausdorff-dimension of $S(c_0, s_0)$ is bounded below by*

$$\dim(S(c_0, s_0)) \geq (n - 1) - \frac{k_0}{s_0}. \quad (1.5)$$

Remark. For $n = 2$, the lower bound in (1.5) can be improved to $1 - \frac{k_0}{s_0 \cdot e^{s_0/2}}$.

Note that (1.5) is trivially satisfied for $n = 1$ and moreover that $S(c_0, s_0)$ is nonempty whenever the lower bound in (1.5) is positive. Thus, combining (1.4) and (1.5) we obtain the following.

Corollary 1.2. *When M has only one cusp, there exists a height $t_0 \geq 0$ such that $[t_0, \infty) \subset S_{\mathcal{H}}$ and positive constants $k_0, k_u > 0$ such that for $t \geq t_0$ we have*

$$(n - 1) - \frac{k_0}{t} \leq \mathfrak{D}_{\mathcal{H}}^0(t) \leq n - \frac{k_u}{t \cdot e^{2nt}}.$$

Finally, note from Section 2.4 that SH_0^+ can be identified with the quotient of \mathbb{R}^n by a discrete cocompact group Γ_∞ acting on \mathbb{R}^n such that the projection map

$$\mathbb{R}^n \ni x \mapsto [x] \equiv v_x \in \mathbb{R}^n / \Gamma_\infty = SH_0^+ \quad (1.6)$$

is surjective and a local isometry. When $c_0 > 0$, the set S_{c_0} of vectors $v \in SH_0^+$ (not necessarily bounded but defined in the same way as above) for which the first height h_1 equals c_0 can be identified with the quotient of $\tilde{S}_{c_0} / \Gamma_\infty$ where $\tilde{S}_{c_0} \subset \mathbb{R}^n$ consists of a countable disjoint union of $(n - 1)$ -dimensional Euclidean spheres in \mathbb{R}^n . As a second theorem, lifting the set of bounded (with respect to e) vectors $S(c_0) \equiv \bigcup_{s_0 \geq 0} S(c_0, s_0)$ with first penetration height c_0 to $\tilde{S}(c_0)$, we show the following.

Theorem 1.3. *When $n \geq 2$, for each sphere S in \tilde{S}_{c_0} we have that $\tilde{S}(c_0) \cap S$ is an absolute winning set in S .*

The theorem also holds if M is pinched negatively curved. For the definition of the absolute winning game we refer to McMullen [12]. Recall that an *absolute winning* set in a submanifold S of \mathbb{R}^n has full Hausdorff-dimension and is in fact *thick* in S ,³ see [3]. Moreover, an absolute winning set in \mathbb{R}^n is preserved under quasi symmetric homeomorphisms and a countable intersection of absolute winning sets is absolute winning.

1.2.2. Avoiding a point. Fix a point x_0 in M which we view as obstacle that is disjoint to a given base point o and let d be the Riemannian distance function on M . Fix a technical constant $t_0 \geq 0$ and define for a given vector $v \in SM_o$ the *distance constant* from the subray $\gamma_v|_{[t_0, \infty)}$ to x_0 by

$$\mathcal{D}(v) \equiv \sup_{t \geq t_0} (-\log(d(\gamma_v(t), x_0))).$$

For a typical vector $v \in SM_o$ we have $\mathcal{D}_{o, x_0}^{t_0}(v) = \infty^4$ and we may call v *bounded* if $\mathcal{D}_{o, x_0}^{t_0}(v) < \infty$. By the author's earlier work [24], the set of bounded vectors is of Hausdorff-dimension n and in fact thick. Define the *distance spectrum* of the data (M, o, x_0, t_0)

³ Recall that a subset Y of a metric space Z is thick if for any nonempty open set $O \subset Z$ we have that $\dim(Y \cap O) = \dim(Z)$.

⁴ This follows from the logarithm law of [11]: for almost all vectors $v \in SM_o$ (with respect to the sphere measure on SM_o) we have $\limsup_{t \rightarrow \infty} \frac{-\log(d(\gamma_v(t), x_0))}{\log(t)} = \frac{1}{n}$.

by $\mathcal{S}_{\mathcal{D}} \equiv \{\mathcal{D}(v) : v \in SM_o \text{ bounded}\} \subset [-\log(t_0 + d(o, x_0)), \infty)$ (for properties of the asymptotic distance spectrum, see Theorem 1.6 below). Define as well the *dimension distance function* $\mathfrak{D}_{\mathcal{D}} : \mathcal{S}_{\mathcal{D}} \rightarrow [0, n]$ by $\mathfrak{D}_{\mathcal{D}}(t) = \dim(\mathcal{B}_{M,o,x_o}^{t_0}(t))$, where

$$\begin{aligned} \mathcal{B}_{M,o,x_o}^{t_0}(t) &\equiv \{v \in SM_o : \mathcal{D}(v) \leq t\} \\ &= \{v \in SM_o : \gamma_v(s) \notin B(x_0, e^{-t}) \text{ for all } s \geq t_0\} \end{aligned}$$

which is the set of rays $\gamma_v|_{[t_0, \infty)}$ avoiding the ball $B(x_0, e^{-t})$.

Our next theorem establishes a Jarník-type inequality as in (1.4) for the obstacle x_0 .

Theorem 1.4. *When M is compact, there exist a time $t_0 \geq 0$, a distance d_0 and positive constants $k_u, k_l > 0$ such that for all $d_0 \leq t \in \mathcal{S}_{\mathcal{D}}$ we have*

$$n - \frac{k_l}{e^{n/2t}} \leq \mathfrak{D}_{\mathcal{D}}(t) \leq n - \frac{k_u}{t \cdot e^{nt}}.$$

Remark. Using the arguments of [26], a Jarník-type inequality can be obtained when M is convex-cocompact, when restricting to positively recurrent vectors in SM_o .

Given a bounded vector $v \in SM_o$ consider the discrete set $\{t_i(v) : i \in \mathbb{N}\} \subset [t_0, \infty)$ of local minima for the distance function $t_0 \leq t \mapsto d(\gamma_v(t), x_0)$ and let $d_i(v) = d(\gamma_v(t_i(v)), x_0)$ be the corresponding distances. Given the parameters $c_0, s_0 \in \mathbb{R}$, define the subset of bounded vectors

$$S(c_0, s_0) \equiv \{v \in SM_o : d_1(v) = e^{-c_0}, d_i(v) \geq e^{-s_0} \text{ for all } i \in \mathbb{N}_{\geq 2}\},$$

which is the set of rays $\gamma_v|_{[t_0, \infty)}$ that have precisely distance e^{-c_0} at time $t_1(v)$ (hence are tangent to the ball $B(x_0, e^{-c_0})$) and avoid $B(x_0, e^{-s_0})$ for all $t > t_1(v)$. Parkkonen, Paulin [15] showed for $n \geq 2$ that $S(c_0, c_0)$ is nonempty for small $c_0 \leq -\log(2)$, assuming a large injectivity radius of M of curvature at most -1 .

Our next theorem deals with large parameters s_0, c_0 and establishes a lower bound on the dimension.

Theorem 1.5. *Let $n \geq 2$. There exists a time $t_0 \geq 0$, a distance $d_0 \in \mathbb{R}$ and a positive constant $k_0 > 0$ and $k_1 \geq 0$, independent of c_0 and s_0 , such that for $c_0 \geq d_0$ and $s_0 \geq 2c_0 + k_1$, the Hausdorff-dimension of $S(c_0, s_0)$ is bounded below by*

$$\dim(S(c_0, s_0)) \geq (n - 1) - \frac{k_0}{s_0}.$$

Remark. Due to the condition that $s_0 \geq 2c_0$, Theorem 1.5 does not guarantee the existence of a Hall ray at the point x_0 (defined as for the case of a cusp). We hope, however, that this condition can be relaxed.

Finally, suppose that M is a complete geometrically finite Riemannian manifold of curvature at most -1 . Define for $v \in SM_o$ also the *asymptotic distance constant*

$$\mathcal{D}^+(v) \equiv \limsup_{t \rightarrow \infty} (-\log(d(\gamma_v(t), x_0))).$$

Let SM_o^+ denote the set of positively recurrent vectors $v \in SM_o$. Denote by $\mathcal{S}_{\mathcal{D}}^+$ the *asymptotic distance spectrum* consisting of the asymptotic distance constants $\mathcal{D}^+(v)$ with $v \in SM_o^+$. As for the asymptotic height and spiraling spectra below, using a result of Maucourant [10], we show the following properties.

Theorem 1.6. *The asymptotic distance spectrum $\mathcal{S}_{\mathcal{D}}^+$ is bounded below and $\mathcal{S}_{\mathcal{D}}^+$ equals the closure of the logarithmic distances $-\log(d(x_0, \alpha))$ from x_0 to closed geodesics α in M ; in particular, we have for the Hurwitz constant $\mathfrak{h}_{\mathcal{D}} \equiv \inf \mathcal{S}_{\mathcal{D}}^+$ that*

$$e^{-\mathfrak{h}_{\mathcal{D}}} = \sup_{\alpha \text{ a closed geodesic in } M} d(x_0, \alpha).$$

Remark. To the best of the author's knowledge, these properties do not already exist in the literature.

1.2.3. Avoiding a closed geodesic. Fix a closed geodesic α_0 in M . For geodesic rays that avoid the obstacle α_0 appropriate neighborhoods of α_0 should in fact be given in the unit tangent bundle SM . We therefore follow [8, 16] and consider the closed ε_0 -neighborhood $\mathcal{N}_{\varepsilon_0}(\alpha_0)$ of α_0 in M where $\varepsilon_0 > 0$ is sufficiently small with respect to α_0 . Bounded penetration lengths of a geodesic γ in $\mathcal{N}_{\varepsilon_0}(\alpha_0)$ (by negative curvature) imply that $\dot{\gamma}$ avoids a neighborhood of $\dot{\alpha}_0$ in SM . More precisely, given a geodesic γ in M , define its *penetration length* at time t by $\mathcal{L}_{\alpha_0, \varepsilon_0}(\gamma, t) = 0$ if $\gamma(t) \notin \mathcal{N}_{\varepsilon_0}(\alpha_0)$ and otherwise by $\mathcal{L}_{\alpha_0, \varepsilon_0}(\gamma, t) \equiv \ell(I)$, where $\ell(I)$ denotes the length of the maximal connected interval $I \subset \mathbb{R}$ such that $t \in I$ and $\gamma(s) \in \mathcal{N}_{\varepsilon_0}(\alpha_0)$ for all $s \in I$.

Fix again a base point $o \in M$ with $o \notin \mathcal{N}_{\varepsilon_0}(\alpha_0)$. Using the terminology from [16], define for $v \in SM_o$ the *spiraling constant* of γ_v in $\mathcal{N}_{\varepsilon_0}(\alpha_0)$ by

$$\mathcal{L}(v) \equiv \sup_{t \geq 0} \mathcal{L}_{\alpha_0, \varepsilon_0}(\gamma_v, t). \quad (1.7)$$

When M is compact, a typical vector $v \in SM_o$ satisfies $\mathcal{L}(v) = \infty^5$ and we call v *bounded* when each possible penetration length in $\mathcal{N}_{\varepsilon_0}(\alpha_0)$ is bounded above by $\mathcal{L}(v) < \infty$. However, we remark that (even for negative curvature or when convex-cocompact) the set of bounded vectors $v \in SM_o$ is of Hausdorff-dimension n and in fact an absolute winning set; see [24], also for further generalizations.

Define the *spiraling spectrum* of the data $(M, o, \alpha_0, \varepsilon_0)$ by $\mathcal{S}_{\mathcal{L}} \equiv \{\mathcal{L}(v) : v \in SM_o \text{ bounded}\} \subset [0, \infty)$.

Remark. Using a different setup, [16] showed that the asymptotic spiraling spectrum⁶ $\mathcal{S}_{\mathcal{L}}^+$ satisfies Properties 1. - 3. above where we replace \mathcal{P} by the set of periodic vectors in SM .

By analogy to (1.3), define the *dimension spiraling functions* $\mathfrak{D}_{\mathcal{L}}, \mathfrak{D}_{\mathcal{L}}^0 : \mathcal{S}_{\mathcal{L}} \rightarrow [0, n]$ on the spectrum $\mathcal{S}_{\mathcal{L}}$. Let $t \in \mathcal{S}_{\mathcal{L}}$ be a given length and denote by

$$\mathcal{B}_{M, o, \alpha_0, \varepsilon_0}(t) \equiv \{v \in SM_o : \mathcal{L}(v) \leq t\}$$

the set of rays γ_v with spiraling constants bounded above by length t such that $\mathfrak{D}_{\mathcal{L}}(t) = \dim(\mathcal{B}_{M, o, \alpha_0, \varepsilon_0}(t))$. From the author's earlier work [26], when M is compact, there exist a length $t_0 \geq 0$ and constants $k_u, k_l > 0$ such that for all $t_0 \leq t \in \mathcal{S}_{\mathcal{L}}$ we have

$$n - \frac{k_l}{t \cdot e^{n/2t}} \leq \mathfrak{D}_{\mathcal{L}}(t) \leq n - \frac{k_u}{t \cdot e^{nt}}. \quad (1.8)$$

Remark. A similar inequality holds when we replace the ε_0 -neighborhood of α_0 by the one of a higher-dimensional (up to codimension one) totally geodesic submanifold which is (ε_0, T) -immersed (see Section 2.1 or [15] for a definition) or for M convex-cocompact; see [26] for further details.

⁵ This follows from the logarithm law of [8]: for almost all vectors $v \in SM_o$ (sphere measure) we have $\limsup_{t \rightarrow \infty} \frac{\mathcal{L}_{\alpha_0, \varepsilon_0}(\gamma_v, t)}{\log(t)} = \frac{1}{n}$.

⁶ That is the spectrum of the asymptotic heights $\mathcal{L}^+(v), v \in SM_o$, where we use the 'limsup' in (1.7).

In the following, we establish nontrivial bounds for $\mathfrak{D}_{\mathcal{L}}^0(t)$. Each bounded vector $v \in SM_o$ determines a sequence of countably many discrete penetration times $t_i(v) \geq 0$ and penetration lengths $l_i(v) = \mathcal{L}_{\alpha_0, \varepsilon_0}(\gamma_v, t_i(v)) > 0$ such that $\gamma_v([t_i(v), t_i(v) + l_i(v)]) \subset \mathcal{N}_{\varepsilon_0}(\alpha_0)$; note that we set $l_i(v) = 0$ for all large i if γ_v eventually avoids $\mathcal{N}_{\varepsilon_0}(\alpha_0)$. Given $c_0 \geq 0$ and $s_0 \geq 0$, define the set of bounded vectors for which the first penetration length $l_1(v)$ equals c_0 and all others are bounded above by s_0 ,

$$S(c_0, s_0) \equiv \{v \in SM_o : l_1(v) = c_0, l_i(v) \leq s_0 \text{ for all } i \in \mathbb{N}_{\geq 2}\}.$$

When $n \geq 2$, Parkkonen, Paulin [15] showed that $S(c_0, \bar{s}_0)$ is nonempty for all sufficiently large lengths $c_0 \geq \bar{c}_0$ and some constant \bar{s}_0 . Note that when $S(c_0, c_0)$ is nonempty for all sufficiently large $c_0 \geq t_0$, this implies the existence of a *Hall ray* at the closed geodesic α_0 , that is, there exists a length $t_0 \geq 0$ such that $[t_0, \infty) \subset \mathcal{S}_{\mathcal{L}}$.

Our next theorem establishes a lower bound on the dimension of this set.

Theorem 1.7. *Let $n \geq 2$. There exists a length $t_0 \geq \log(2)$ and a positive constant $k_0 > 0$, independent of s_0 and c_0 , such that for all lengths $c_0, s_0 \geq t_0$, the Hausdorff-dimension of $S(c_0, s_0)$ is bounded below by*

$$\dim(S(c_0, s_0)) \geq (n - 1) - \frac{k_0}{s_0}. \quad (1.9)$$

Remark. A similar lower bound holds when we replace the ε_0 -neighborhood of α_0 by the one of a higher-dimensional (up to codimension 2) totally geodesic submanifold which is (ε_0, T) -immersed.

Combining (1.8) and (1.9) we obtain the following corollary.

Corollary 1.8. *Let M be compact. There exists a length $t_0 \geq 0$ such that $[t_0, \infty) \subset \mathcal{S}_{\mathcal{L}}$ and positive constants $k_0, k_u > 0$ such that for $t \geq t_0$ we have*

$$(n - 1) - \frac{k_0}{t} \leq \mathfrak{D}_{\mathcal{L}}^0(t) \leq n - \frac{k_u}{t \cdot e^{nt}}.$$

1.3. Applications to Diophantine approximation and further discussion. We will now shortly discuss applications of Sections 1.2.1 and 1.2.3 to Diophantine approximation. For further background and details, we refer to [5, 15] and references therein.

1.3.1. Imaginary quadratic number fields. For a positive square free integer d let \mathcal{O}_d be the ring of integers in the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$. For a complex number $z \in \mathbb{C}$, define its *approximation constant* by

$$c_d(z) \equiv \inf_{(p,q) \in \mathcal{O}_d \times (\mathcal{O}_d \setminus \{0\})} |q|^2 |z - \frac{p}{q}|. \quad (1.10)$$

When $c_d(z) > 0$ we call z *badly approximable* by $\mathbb{Q}(i\sqrt{d})$ and denote $\mathbf{Bad}_d \equiv \{z \in \mathbb{C} : c_d(z) > 0\}$ the set of badly approximable complex numbers. Let \mathcal{S}_d be the *spectrum* of logarithmic approximation constants $-\log(c_d(z))$, $z \in \mathbf{Bad}_d$. Define the dimension functions $\mathfrak{D}_d, \mathfrak{D}_d^0 : \mathcal{S}_d \rightarrow [0, 2]$ as in (1.3).

Remark. The asymptotic spectrum⁷ \mathcal{S}_d^+ is bounded below, contains a Hall ray and $\mathcal{S}_d^+ \cap \mathbb{R}$ equals the closure of the logarithmic approximation constants of $z \in \mathbb{C} - \mathbb{Q}(i\sqrt{d})$ quadratic over $\mathbb{Q}(i\sqrt{d})$ (see [10, 15, 22]).

⁷ That is the spectrum of logarithmic approximation constants $-\log(c_d^+(z))$, $z \in \mathbb{C}$, where we use the ‘liminf’ in (1.10).

Let \mathcal{I}_d be the ideal class group of $\mathbb{Q}(i\sqrt{d})$ which contains only one ideal class if

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}. \quad (1.11)$$

Theorem 1.9. *Let d be as in (1.11). The sets \mathbf{Bad}_d in \mathbb{C} as well as $\mathbf{Bad}_d \cap S(p/q, r)$ in $S(p/q, r)$ are absolute winning, where $S(p/q, r) = \partial B(p/q, r)$ with $(p, q) \in \mathcal{O}_d^2$, $q \neq 0$ and $r \leq \frac{k}{2|q|^2}$. Moreover, there exist t_0 and t_1 such that $[t_0, \infty) \subset \mathcal{S}_d$ and positive constants $k_0, k_u > 0$ such that for $t \geq t_0$ we have*

$$1 - \frac{k_0}{t \cdot e^{t/2}} \leq \mathfrak{D}_d^0(t) \leq 2 - \frac{k_u}{t \cdot e^{4t}},$$

as well as for $t_1 \leq t \in \mathcal{S}_d$

$$2 - \frac{k_0}{t \cdot e^t} \leq \mathfrak{D}_d(t) \leq 2 - \frac{k_u}{t \cdot e^{4t}}.$$

Proof. The group $G = PSL(2, \mathbb{C})$ acts on the real hyperbolic upper half space \mathbb{H}^3 (as a subset of \mathbb{C}^2) as the full group of orientation preserving isometries and restricted to $\partial_\infty \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ by Möbius transformations. Moreover, the *Bianchi group* $\Gamma_d \equiv PSL(2, \mathcal{O}_d)$ is a lattice in G , possibly with torsion, so that $M_d = \mathbb{H}^3/\Gamma_d$ is a finite volume hyperbolic orbifold (note however that the previous results are still valid in this case). The finitely many cusps in M_d are in bijective correspondence to the ideal classes in \mathcal{I}_d . Thus, by assumption, let e be the only cusp in M_d and consider the setting of Section 1.2.1. For the map $\mathbb{C} \ni z \mapsto v_z \in SH_0^+$ given in (1.6) we obtain the following correspondence.

Lemma 1.10. *We have $\mathcal{H}(v_z) = -\log(k \cdot c_d(z))$, where $k > 0$ depends on the height function β_e .*

The proof of Lemma 1.10 follows in a similar way to the one in [14] (see also [10, 22] again), using Lemma 2.1 below and that $PSL(2, \mathcal{O}_d) \cdot \infty = \mathbb{Q}(i\sqrt{d}) \cup \{\infty\}$. Applying the results from Section 1.2.1 finishes the proof. \square

Remark. The above theorem holds without the restriction that d is as in (1.11), that is, when M_d has several cusps. This follows along the lines of the respective proofs in [24, 26] and Section 2.3 below, where for (1.4) we need to replace bounded with respect to one cusp by *uniformly bounded* in [26], that is bounded with respect to all cusps with the same height.

1.3.2. Quadratic irrational numbers. This section closely follows [8, 16] to which we also refer for further details. Let either $K = \mathbb{Q} \subset \mathbb{R} = \hat{K}$ or $K = \mathbb{Q}(i\sqrt{d}) \subset \mathbb{C} = \hat{K}$, where $d \in \mathbb{N}$ is square-free. Denote by \mathcal{O}_K the ring of integers of K , that is \mathbb{Z} or \mathcal{O}_d , and by K_{quad} the real quadratic irrational (complex) numbers over K in \hat{K} . For $\beta \in K_{quad}$, let $\beta^\sigma \in K_{quad}$ be its Galois conjugate. The subgroup $PSL(2, \mathcal{O}_K)$ of $PSL(2, \hat{K})$, acting by Möbius transformations on $\hat{K} \cup \{\infty\}$, preserves K as well as K_{quad} and $\psi(\beta^\sigma) = (\psi(\beta))^\sigma$ for all $\beta \in K_{quad}$, $\psi \in PSL(2, \mathcal{O}_K)$.

Fix $\beta_0 \in K_{quad}$ and let $\mathcal{P}_{\beta_0} = PSL(2, \mathcal{O}_K) \cdot \{\beta_0, \beta_0^\sigma\}$ be its orbit in \hat{K} which is dense and countable. For $x \in \hat{K}$ define its *approximation constant* by

$$c_{\beta_0}(x) \equiv \inf_{(\beta, \beta^\sigma) \in \mathcal{P}_{\beta_0}} |\beta - \beta^\sigma|^{-1} |x - \beta|,$$

which determines the set $\mathbf{Bad}_{\hat{K}, \beta_0} \equiv \{x \in \hat{K} : c_{\beta_0}(x) > 0\}$ and the spectrum $\mathcal{S}_{\hat{K}, \beta_0} \equiv \{-\log(c_{\beta_0}(x)) : x \in \mathbf{Bad}_{\hat{K}, \beta_0}\}$; for properties of the asymptotic spectrum $\mathcal{S}_{\hat{K}, \beta_0}^+$ see [16]. Define the dimension function $\mathfrak{D}_{\hat{K}, \beta_0} : \mathcal{S}_{\hat{K}, \beta_0} \rightarrow [0, 1]$ as in (1.3).

Theorem 1.11. *Bad $_{\hat{K},\beta_0}$ is absolute winning in \hat{K} . Moreover, there exist t_0 and t_1 such that $[t_0, \infty) \subset \mathcal{S}_{\mathbb{C},\beta_0}$ and positive constants $k_0, k_u > 0$ such that for $t \geq t_0$ we have*

$$\mathfrak{D}_{\mathbb{C},\beta_0}^0(t) \geq 1 - \frac{k_0}{t},$$

as well as for $t_1 \leq t \in \mathcal{S}_{\hat{K},\beta_0}$

$$\mathfrak{D}_{\hat{K},\beta_0}(t) \geq \dim(\hat{K}) - \frac{k_0}{t \cdot e^{\dim(\hat{K})t/2}}.$$

Sketch of the proof. For details of the following we refer to [16]. Note that \mathcal{P}_{β_0} determines a unique closed geodesic α_0 in the modular surface in $M = \mathbb{H}^2/PSL(2, \mathbb{Z})$, respectively in the Bianchi orbifold $M = \mathbb{H}^3/PSL(2, \mathcal{O}_d)$. Moreover, each pair $(\beta, \beta^\sigma) \in \mathcal{P}_{\beta_0}$ corresponds to a unique lift of α_0 . Fix one of the cusps e in M . Moreover, we replace the base point o with a sufficiently small cusp neighborhood H_0 of e disjoint to $\mathcal{N}_{\varepsilon_0}(\alpha_0)$. Note that the results in Section 1.2.3 are still valid if we replace SM_o with SH_0^+ and restrict to a compact fundamental domain F in \hat{K} for the action of $\Gamma_\infty = \text{Stab}_{PSL(2, \mathcal{O}_K)}(\infty)$ (there is a constant c_0 such that geodesic rays $\gamma_{\tilde{o},x}$ starting in a given point \tilde{o} and $\gamma_{\tilde{H}_0,x}$ starting orthogonally to a horoball \tilde{H}_0 based at ∞ in \mathbb{H}^2 or \mathbb{H}^3 and ending at the same point in $x \in F$ project to geodesic rays in M whose spiraling lengths differs at most by c_0). As in Lemma 1.10 (using Lemma 2.1 below), for the map $F \ni x \mapsto v_x \in SH_0^+$ given in (1.6) there is a constant c_1 such that $|\mathcal{L}(v_x) + \log(c_{\beta_0}(x))| \leq c_1$. Remarking that the compactness in (1.8) was only required for the upper bound (see [26], Section 3.3), we finish the proof by applying the results of Section 1.2.3. \square

1.3.3. Further applications. Considering concrete lattices of the real (or complex) hyperbolic space, further arithmetic applications can be obtained from results concerning the dynamics of the geodesic flow on the corresponding orbifold. For instance, for results of approximation of real Hamiltonian quaternions or of elements of a real Heisenberg group by 'rational elements' we refer to [14, 15] (avoiding a cusp), and for their approximation by 'quadratic irrational elements' we refer to [8, 16] (avoiding a closed geodesic).

1.3.4. Some discussion. We conclude this section by further questions and remarks.

1. Can we determine additional properties of the above dimension functions \mathfrak{D} and \mathfrak{D}^0 ? For instance, the obtained results do not affect the question of whether or not \mathfrak{D} and \mathfrak{D}^0 are continuous on some subintervals of the spectra.
In addition, it is not clear whether \mathfrak{D} and \mathfrak{D}^0 are also positive outside of the determined intervals $[t_0, \infty)$. Respectively, what is the value of t_0 ?
2. The determined bounds for \mathfrak{D} and \mathfrak{D}^0 are of an asymptotic flavor. Moreover, they do not match and further effort for more precise bounds is necessary, in particular for \mathfrak{D}^0 .
3. One may also study the dimension function $\mathfrak{D}^1(t)$ defined as the Hausdorff dimension of elements x with approximation constant $t \leq c(x) < \infty$. Moreover, as remarked earlier, (for each of the above dimension functions) a lower bound for \mathfrak{D} gives a lower bound for \mathfrak{D}^+ , the asymptotic dimension function. However, it seems to be hard to determine an upper bound for \mathfrak{D}^+ .
4. As remarked above, several of the properties of the asymptotic spectra (height, distance and spiraling) rely on a result of [10] for negatively curved manifolds. The crucial tool in [10] is Anosov's closing lemma. Using a 'metric version' of the closing lemma in the context of proper geodesic CAT(-1) metric spaces (see

[25]) we are able to show the denseness of approximation constants corresponding to periodic elements in the asymptotic spectrum (height, distance, spiraling) in a more general setting. This might have further applications, for instance to groups acting on metric trees.

2. PROOFS

Recall that (in most cases) we restricted to constant negative curvature and considered only finite volume hyperbolic manifolds $M = \mathbb{H}^{n+1}/\Gamma$. The main reason for these restrictions is that, using Lemma 2.1 below, we can relate our setup to a Diophantine setting on the full boundary $S^n = \mathbb{R}^n \cup \{\infty\}$ at infinity of \mathbb{H}^{n+1} which provides the existence of suitable measures and can be partitioned in a nice way. In particular we obtain the setup and the requirements from our earlier work [26] and can apply its formalism in order to determine nontrivial bounds on the Hausdorff-dimension.

We begin in Section 2.1 with preliminaries and prove Theorem 1.6 in Section 2.2. The setup of [26], Section 2, is recalled in Section 2.3 which we already adopt to our setting. We then prove Theorem 1.3 in Section 2.4, as well as the Theorems 1.1, 1.5, 1.7 in Section 2.5 and finally Theorem 1.4 in Section 2.6.

2.1. Some background and notation in hyperbolic geometry. A reference for further details and definitions of the following is given by [2, 23]. Let \mathbb{H}^n be the n -dimensional real hyperbolic upper half-space model where d denotes the hyperbolic distance on \mathbb{H}^n . Assume all geodesic segments, rays or lines to be parametrized by arc length and identify their images with their point sets in \mathbb{H}^n . For a noncompact convex subset $Y \subset \mathbb{H}^n$, let $\partial_\infty Y$ denote its visual boundary, that is, the set of equivalence classes of asymptotic rays in Y . Identify $\partial_\infty \mathbb{H}^n$ with the set $S^{n-1} \cong \mathbb{R}^{n-1} \cup \{\infty\}$ and equip $\bar{\mathbb{H}}^n = \mathbb{H}^n \cup S^{n-1}$ with the cone topology. If γ is a ray in \mathbb{H}^n we will simply write $\gamma(\infty)$ for the corresponding point in $\partial_\infty \mathbb{H}^n$. For any two points p and q in $\bar{\mathbb{H}}^n$ denote by $\gamma_{p,q}$ the geodesic segment, ray or line in \mathbb{H}^n connecting p and q . Given $\xi \in \partial_\infty \mathbb{H}^{n+1}$ and $y \in \mathbb{H}^{n+1}$, the *Busemann function* $\beta = \beta_{\xi,y} : \mathbb{H}^n \rightarrow \mathbb{R}$ is defined by

$$\beta(x) \equiv \lim_{t \rightarrow \infty} d(y, \gamma_{y,\xi}(t)) - d(x, \gamma_{y,\xi}(t)),$$

which (exists and) is continuous and convex on \mathbb{H}^n and $\beta(y) = 0$. The sublevel sets $H_t \equiv \beta^{-1}([t, \infty))$ of β are called *horoballs* at ξ (with respect to y). If $\xi = \infty$, then H_t equals $\mathbb{R}^{n-1} \times [s, \infty)$ for some $s > 0$, and if $\xi \in \mathbb{R}^{n-1}$, then H_t equals an Euclidean ball based at ξ . Given a horoball $C \subset \mathbb{H}^n$ based at $\partial_\infty C$ we can associate a Busemann function, denoted by β_C and parametrized such that $\beta_C^{-1}([0, \infty)) = C$. For three points $o, x, y \in \bar{\mathbb{H}}^n$, let $(x, y)_o$ denote the *Gromov-product* at o and for $\xi, \eta \in \partial_\infty \mathbb{H}^n$, let

$$(\xi, \eta)_o \equiv \lim_{t \rightarrow \infty} (\gamma_{o,\xi}(t), \gamma_{o,\eta}(t))_o$$

be the extended Gromov-product at o . Define the *visual metric* at $o \in \mathbb{H}^n$ by $d_o : \partial_\infty \mathbb{H}^n \times \partial_\infty \mathbb{H}^n \rightarrow [0, \infty)$ by $d_o(\xi, \xi) \equiv 0$ and for $\xi \neq \eta$ by

$$d_o(\xi, \eta) \equiv e^{-(\xi, \eta)_o}.$$

Then $(\partial_\infty \mathbb{H}^n, d_o)$ is a compact metric space. Note that the visual metric at a point $o \in \mathbb{H}^n$ is (locally) bi-Lipschitz equivalent to the Euclidean metric d_E on \mathbb{R}^{n-1} : for every compact subset $K \subset \mathbb{R}^n$, there exists a constant $c_K > 0$ such that for all $\xi, \eta \in K$,

$$c_K^{-1} d_o(\xi, \eta) \leq d_E(\xi, \eta) \leq c_K d_o(\xi, \eta); \quad (2.1)$$

see [8], Lemma 2.3.

Now let M be a $(n + 1)$ -dimensional finite volume hyperbolic manifold. Then there is a discrete, torsion-free subgroup Γ of the isometry group of \mathbb{H}^{n+1} identified with the (free) fundamental group $\pi_1(M)$ of M acting on \mathbb{H}^{n+1} such that the manifold \mathbb{H}^n/Γ with the induced smooth and metric structure is isometric to M . Let $\bar{\pi} : \mathbb{H}^n \rightarrow \mathbb{H}^n/\Gamma \cong M$ be the projection or covering map. When Γ is a non-elementary geometrically finite discrete group, then we call M geometrically finite; see [1].

A sufficiently small cusp neighborhood of M lifts to a Γ -invariant countable collection \mathcal{C} of precisely invariant horoballs in \mathbb{H}^n . In particular, these horoballs are pairwise disjoint. Note that by Γ -invariance we have $\gamma \circ \beta_C = \beta_{\gamma(C)}$ for every $\gamma \in \Gamma$ and $C \in \mathcal{C}$; in particular, each β_C projects to β_e on M as in the introduction.

A closed geodesic α in M lifts to a Γ -invariant countable collection \mathcal{C} of geodesic lines in \mathbb{H}^{n+1} which is (ε, T) -immersed (using the terminology of [8, 15]), that is, given $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that for any two distinct lines C_1 and C_2 in \mathcal{C} we have that the diameter

$$\text{diam}(\mathcal{N}_\varepsilon(C_1) \cap \mathcal{N}_\varepsilon(C_2)) \leq T;$$

here and hereafter, $\mathcal{N}_\varepsilon(S)$ denotes the closed ε -neighborhood of a set S in a metric space.

A point x in M lifts to a Γ -invariant countable collection \mathcal{C} of points which are τ_0 -separated and, if M is compact, R_0 -spanning for some $\tau_0 > 0$ and $R_0 > 0$; that is, for any distinct points $z, y \in \mathcal{C}$ we have $d(z, y) \geq \tau_0$ and for any point $z \in \mathbb{H}^n$ there is $y \in \mathcal{C}^3$ such that $d(z, y) \leq R_0$.

Note that for each of these collections, given a point $o \in \mathbb{H}^n$, the set $\{d(o, C) : C \in \mathcal{C}\} \subset \mathbb{R}$ is discrete and unbounded. Finally, note that the dynamics of a geodesic ray in M , in terms of penetration properties of a cusp neighborhood or a neighborhood of a point and a closed geodesic, corresponds to the dynamics of its lift to \mathbb{H}^n in the collection \mathcal{C} as above.

2.2. Proof of Theorem 1.6 (Properties of the asymptotic distance spectrum). In order to avoid giving further definitions and background, we only sketch the proof and refer to [1] for proper definitions and further details. Recall that M is a complete geometrically finite Riemannian manifold of curvature at most -1 . Let CM denote the *convex core* of M which is a closed convex subset of M that can be decomposed into a compact subset K and, unless M is convex-cocompact, into a disjoint union of open sets V_i (where $V_i = \tilde{V}_i \cap CM$ and each \tilde{V}_i determines a disjoint collection of horoballs in the universal cover of M). Let D_0 be the diameter of K , $x \in K$, and γ be a geodesic ray (not necessarily starting in o). If γ is positively recurrent, then it is eventually contained in $\mathcal{N}_{D_0}(CM)$ and it follows from the decomposition that it must intersect the compact set $B(x, 2D_0) \supset K$ infinitely often. This implies that the asymptotic distance spectrum is bounded below by

$$-\log(d(x_0, x) + 2D_0).$$

Recall that $\pi : SM \rightarrow M$ denotes the footpoint projection, set $d_{x_0}(z) \equiv d(x_0, z)$ and define

$$f \equiv (-\log) \circ d_{x_0} \circ \pi : SM \rightarrow \mathbb{R}.$$

Clearly, both π and d_{x_0} are continuous and proper functions so that the same is true for f . Let $\mathcal{P} \subset SM$ denote the set of unit vectors tangent to closed geodesics in M . It follows from [10], Theorem 2, that

$$\mathbb{R} \cap \{\limsup_{t \rightarrow \infty} f(\phi^t(v)) : v \in SM\} = \overline{\{\max_{t \in \mathbb{R}} f(\phi^t(w)) : w \in \mathcal{P}\}}. \quad (2.2)$$

Note that $\limsup_{t \rightarrow \infty} f(\phi^t(v))$ depends only on the asymptotic class of γ_v , hence not on the base point, and we may in fact replace SM in the left hand side of (2.2) with SM_o . Finally, by definition $\limsup_{t \rightarrow \infty} f(\phi^t(v)) = \mathcal{D}^+(v)$ as well as

$$\max_{t \in \mathbb{R}} f(\phi^t(w)) = -\left(\min_{t \in \mathbb{R}} \log(d(x_0, \alpha(t)))\right) \equiv -(\log(d(x_0, \alpha))),$$

where α denotes the closed geodesic determined by w . This finishes the proof.

2.3. The family of resonant sets and a bound for the Hausdorff-dimension. We first introduce our setting which, as remarked in the previous Section 2.1, is for instance satisfied when lifting our setup from Section 1.2 (as stated above). Consider three nonempty countable collections \mathcal{C}^i , $i = 1, 2, 3$, of closed convex sets in \mathbb{H}^{n+1} , where

1. \mathcal{C}^1 is a collection of pairwise disjoint horoballs
2. \mathcal{C}^2 is a collection of (ε_0, T) -immersed geodesic lines,
3. \mathcal{C}^3 is a collection of points which is τ_0 -separated and R_0 -spanning.

Remark. Note that if \mathcal{C}^2 is (ε_0, T) -immersed then it is also (δ, L) -immersed for $\delta > 0$ and $L = L(T, \varepsilon_0, \delta)$. In addition, using a result of [19] about the Patterson-Sullivan measure of a non-elementary convex cocompact Kleinian group and similarly to [26], the collection \mathcal{C}^2 may be replaced by a (ε_0, T) -immersed collection of totally geodesic up to $(n-2)$ -dimensional submanifolds of \mathbb{H}^{n+1} . Moreover, the assumption that \mathcal{C}^3 is τ_0 -separated is only needed for the following lower bounds on the Hausdorff-dimension and the upper bound uses that it is R_0 -spanning.

Fix a base point $o \in \mathbb{H}^{n+1} \cup S^n$. In the first case, we fix a horoball $C_o^1 \in \mathcal{C}^1$ (which we assume to be) based at $\infty \in S^n$ of Euclidean height 1 and exclude it from the collection. Then the Hamenstädt-metric $d_{C_o^1}$ on $\mathbb{R}^{n-1} = \partial_\infty \mathbb{H}^n - \{\infty\} \equiv \bar{X}_1$ with respect to C_o^1 equals the Euclidean metric; by abuse of notation, we write d_o for $d_{C_o^1}$ and $\gamma_{o,\xi}$ for a vertical ray starting on C_o^1 and ending at $\xi \in \mathbb{R}^n$. For the second and third case we let $o = C_o^i \in \mathbb{H}^{n+1}$ such that $o \notin \cup_{C \in \mathcal{C}^2} \mathcal{N}_{\varepsilon_0}(C)$ or $o \notin \cup_{C \in \mathcal{C}^3} B_{\varepsilon_0}(C)$ for some $\varepsilon_0 > 0$ respectively. Let d_o be the visual metric at o on the sphere $S^{n-1} \equiv \bar{X}_i$.

Each collection determines a set of *sizes*

$$\{s_C^i \equiv d(C_o^i, C) : C \in \mathcal{C}^i\} \subset \mathbb{R}_{\geq 0}$$

which we assume to be discrete and unbounded. For a point $x \in \mathbb{H}^n$, distinct to o , we let $\gamma_{o,x}(\infty) \in \partial_\infty \mathbb{H}^{n+1} = S^n$ be the boundary projection of x with respect to o ; by abuse of notation, we write $\partial_\infty x \equiv \gamma_{o,x}(\infty)$ in the following. Then, each collection determines a nonempty collection \mathcal{C}_∞^i at infinity, where

$$\mathcal{C}_\infty^i \equiv \{\partial_\infty C \subset \bar{X}_i : C \in \mathcal{C}^i\} \subset \partial_\infty \mathbb{H}^n = S^{n-1}$$

is the collection of tangency points of the horoballs in \mathcal{C}^1 , endpoints of the geodesic lines in \mathcal{C}^2 or boundary projections of points in \mathcal{C}^3 with respect to o , respectively.

The following Lemma is crucial and relates the 'metric' properties of a point $\xi \in \bar{X}_i$ with respect to the collection \mathcal{C}_∞^i at infinity with the (dynamical) penetration properties of the ray $\gamma_{o,\xi}$ in the collection \mathcal{C}^i .

Lemma 2.1. *There are universal constants $\kappa_u \geq \kappa_l > 0$ and $\bar{c}_0 \geq 0$ with the following property. Let $\gamma = \gamma_{o,\xi}$ be a geodesic line (or ray) starting from o and let $C \in \mathcal{C}^i$ with $\beta_C(\gamma) \geq 0$, $L(\gamma \cap \mathcal{N}_{\varepsilon_0}(C)) \geq c_0$ or, for the third case that $d(o, C) \geq c_0$ and $d(\gamma, C) \leq e^{-c_0}$. Then*

1. $d_o(\xi, \partial_\infty C) = \frac{1}{2}e^{-c}e^{-d(C_o^i, C)}$ if and only if the height $\max_{t \in \mathbb{R}} \beta_C(\gamma(t)) = c$;

2. $\kappa_l e^{-L(\gamma \cap \mathcal{N}_{\varepsilon_0}(C))} \cdot e^{-d(o,C)} \leq d_o(\xi, \partial_\infty C) \leq \kappa_u e^{-L(\gamma \cap \mathcal{N}_{\varepsilon_0}(C))} \cdot e^{-d(o,C)};$
3. $\kappa_l d(\gamma_{o,\xi}, C) \cdot e^{-d(o,C)} \leq d_o(\xi, \partial_\infty C) \leq \kappa_u d(\gamma_{o,\xi}, C) \cdot e^{-d(o,C)}.$

Proof. The first part is an exercise in hyperbolic geometry. The second and third part follow along the lines of the proof of Lemma 3.11 in [24] (see also [6, 8]), which is stated in a slightly different way. \square

We now introduce the conditions and the framework of [26], which is slightly different and already adopted to our setting, required for the lower bound. When there is no need to distinguish between the particular cases, we will omit the index i in the following and consider with \mathcal{C} a collection as above. Define a one-parameter family \mathcal{R} consisting of the *resonant sets* $\mathcal{R}(t)$ of size $s_t \equiv t \geq 0$ by

$$\mathcal{R}(t) \equiv \{\partial_\infty C : C \in \mathcal{C} \text{ such that } s_C = d(C_o, C) \leq t\} \subset \bar{X}. \quad (2.3)$$

In the language of [26], for a closed subset $X \subset \bar{X}$, we obtain the set of *badly approximable points* in X (with respect to the family \mathcal{R}) with approximation constant at least e^{-c} , $c < \infty$, which is given by

$$\mathbf{Bad}_X(\mathcal{R}, c) \equiv \{\xi \in X : d_o(\xi, \partial_\infty C) \geq e^{-c} e^{-d(C_o, C)} \text{ for all } C \in \mathcal{C}\}.$$

Given X and a technical parameter t_* , needed below, we determine the parameter space (Ω, ψ) as follows. Define (for the respective metrics) the monotonic function ψ on the set of *formal balls* $\Omega \equiv X \times [t_*, \infty)$ by

$$\psi(\xi, t) \equiv B_{d_o}(\xi, e^{-t}) \cap X, \quad (\xi, t) \in \Omega,$$

which is the restriction of the monotonic function $\bar{\psi}(x, t) \equiv B_{d_o}(x, e^{-t}) \subset \bar{X}$, $(x, t) \in \bar{X} \times \mathbb{R}^+$ to Ω . Note that we have, $\text{diam}(\psi(x, t)) \leq 2e^{-t}$. Moreover, for $c \geq \log(2)$ consider the following constants

$$d_* = \log(3), \quad d_c = -\log(1 - e^{-c}) \leq \log(2),$$

for which we remark that, since the resonant set $\mathcal{R}(t)$ is discrete for all $t \geq t_*$, it follows for all $\xi \in X$ that

$$\xi \notin \mathcal{N}(\mathcal{R}(t), e^{-t}) \implies B_{d_o}(\xi, e^{-(t+d_*)}) \cap \mathcal{N}_{d_o}(\mathcal{R}(t), e^{-(t+d_*)}) = \emptyset,$$

in particular this holds for $\psi(\xi, t + d_*)$ replaced by $B_{d_o}(\xi, e^{-(t+d_*)})$; here, $\mathcal{N}(\mathcal{R}(t), r)$ denotes the closed r -neighborhood of the set $\mathcal{R}(t)$ in \bar{X} with respect to the metric d_o .

Fix $l_* \in \mathbb{R}$ and consider the following conditions, given the parameter $c > 0$.

(S0) There exists a formal ball $\omega_1 \equiv (x, t_*) \in \Omega$ such that

$$\psi(\omega_1) \subset X \cap \left(\bigcup_{C \in \mathcal{C}^i: s_C \leq t_* - l_* - c} \mathcal{N}_{d_o}(\partial_\infty C, e^{-(s_C + 2c + l_*)}) \right)^C$$

Let μ be a locally finite Borel measure on \bar{X} .

($\mu 1$) (Ω, ψ, μ) satisfies a *power law with respect to the parameters* (τ, c_l, c_u) , where $\tau > 0$, $c_u \geq c_l > 0$, that is, we have $\text{supp}(\mu) = X$ and

$$c_l e^{-\tau t} \leq \mu(\psi(x, t)) \leq c_u e^{-\tau t}$$

for all formal balls $(x, t) \in \Omega$.

($\mu 2$) (Ω, ψ, μ) is called τ_c -decaying with respect to \mathcal{R} , if all formal balls $\omega = (\xi, t + d_c) \in \Omega$ we have

$$\mu(\psi(\omega) \cap \mathcal{N}_{d_o}(\mathcal{R}(t - l_*, c), e^{-(t+c-d_*)})) \leq \tau_c \cdot \mu(\psi(\omega)), \quad (2.4)$$

where $\tau_c < 1$ is a constant depending on c ; here, $\mathcal{R}(s, c)$ denotes the resonant sets with sizes in the window $(s - c, s]$, that is

$$\mathcal{R}(s, c) \equiv \mathcal{R}(s) - \mathcal{R}(s - c) = \{\partial_\infty C \in \mathcal{C}_\infty : s - c \leq d(C_o, C) \leq s\}.$$

Remark. Condition (S0) is trivially satisfied for all $(\xi, t_*) \in \Omega$ whenever $c \geq t_* - \min\{s_C : C \in \mathcal{C}\}$. Note that condition ($\mu 1$) reflects how well a ball in X can be separated into smaller balls of the same radius and could be stated in different terms.

These conditions imply the following lower bound.

Proposition 2.2. *Under these conditions and in our setting we have*

$$\dim(\mathbf{Bad}_X(\mathcal{R}, 2c + l_*)) \geq \tau - \frac{\log(2c_u^2 c_l^{-2} e^{\tau(d_* + d_c)}) + |\log(1 - \tau_c)|}{c}.$$

Remark. In the case that $X = \mathbb{R}^n$, a more precise lower bound can be determined; see below for the Jarník-type inequality. Moreover, under a condition converse to (2.4) a similar upper bound is given in [26].

Proof. Starting with the formal ball $\omega_1 \in \Omega$ given from (S0) and with $t_1 = t_*$, we apply the formalism of [26], Section 2.2, in order to inductively construct a ‘strongly treelike collection of subcoverings’ with limit set $A_\infty \subset \psi(\omega_1)$. Conditions ($\mu 1$), ($\mu 2$) thereby establish the inductive steps of the construction and using inequality (2.18) of [26] we obtain the desired lower bound on the Hausdorff-dimension of A_∞ which is in our setting exactly the one given above. Moreover, by Lemma 2.4 in [26] (or rather its proof), every $\xi \in A_\infty$ satisfies $d_o(\xi, \partial_\infty C) \geq e^{-(s_C + 2c + l_*)}$ for every $C \in \mathcal{C}$ with $s_C \geq t_* - c - l_*$. Since also $A_\infty \subset \psi(\omega_1)$ we have by (S0) that $A_\infty \subset \mathbf{Bad}_X(\mathcal{F}, 2c + l_*)$, finishing the proof. \square

In order to show conditions (S0) and ($\mu 2$), we need the following result on the distribution of the sets in the collections \mathcal{C}_∞^i in $\partial_\infty \mathbb{H}^{n+1}$.

Proposition 2.3. *Let $l_1 \equiv -\log(2)$ and $l_2 \geq T + \varepsilon_0$ be sufficiently large. Then, for the Cases 1, 2, we have for distinct sets $C, \bar{C} \in \mathcal{C}^i$ that*

$$d_o(\partial_\infty C, \partial_\infty \bar{C}) \geq e^{-l_i} \cdot e^{-\max\{s_C^i, s_{\bar{C}}^i\}}, \quad (2.5)$$

Moreover, there exists a constant $k_0 = k_0(\tau_0)$ such that, for every $c > 0$ and every ball $B = B_{d_o}(\eta, 2e^{-t})$ with $t \geq 0$,

$$|\{\partial_\infty x \in B : x \in \mathcal{C}^3 \text{ with } s_x = d(o, x) \in (t - c, t]\}| \leq k_0 \cdot c. \quad (2.6)$$

On the other hand, there exist a time $t_3 \geq 0$ and a constant $u_* = u_*(R_0) \geq 0$ such that for every ball $B = B_{d_o}(\eta, e^{-(t-u_*)})$ with $t \geq t_3$ sufficiently large there exists $x \in \mathcal{C}^3$ with $t - u_* \leq d(o, x) \leq t$ and $\partial_\infty x \in B$.

Proof. For the first case, recall that each $C \in \mathcal{C}^1$ is a Euclidean ball in $\mathbb{R}^n \times \mathbb{R}^+$, tangent to a point $\eta \in \mathbb{R}^n$ and of Euclidean radius $r_\eta = e^{-d(C_o^1, C)}/2$. Hence, consider two such disjoint Euclidean balls, tangent to two distinct points $\eta \neq \bar{\eta}$ in \mathbb{R}^n and with Euclidean radius $r_\eta, r_{\bar{\eta}}$. A simple Euclidean computation (Pythagoras) shows that

$$d_{C_o^1}(\eta, \bar{\eta}) = |\eta - \bar{\eta}| \geq 2\sqrt{r_\eta r_{\bar{\eta}}} \geq 2\min\{r_\eta, r_{\bar{\eta}}\} = e^{-l_i} \cdot e^{-\max\{d_i, d_j\}}.$$

The second case follows from Proposition 3.12 in [24].

The same is true for the first part of the third case. Since we will make use of this part twice we recall it for the sake of completeness. For a subset $M \subset S^n$ and $0 \leq a \leq \bar{a}$, consider the truncated cone of M with respect to o ,

$$M(a, \bar{a}) \equiv \{\gamma_{o,\xi}(t) \in \mathbb{H}^{n+1} : \xi \in M, a \leq t \leq \bar{a}\}.$$

Fix $c > 0$, a ball $B = B_{d_o}(\eta, 2e^{-t})$ and note that a point x_∞ with $t - c < d(o, x) \leq t$ lies in B if and only if $x \in B(t - c, t)$. It therefore suffices to estimate the number of $x \in B(t - c, t) \cap \mathcal{C}^3$ which we denote by $G(\eta, t, c)$.

First, we claim that $B(t - c, t)$ is contained in the $(\delta_0 + 2\log(2))$ -neighborhood of the geodesic segment $\gamma_{o,\eta}((t - c, t])$, where δ_0 denotes the hyperbolicity constant of \mathbb{H}^n . To see this, note that for the Gromov-product for $\xi \in B$ and η at o

$$(\xi, \eta)_o \geq -\log(d_o(\xi, \eta)) \geq t - \log(2)$$

and hence, see [2], $d(\gamma_{o,\xi}(s), \gamma_{o,\eta}(s)) \leq \delta_0$, for all $s \leq t - \log(2)$. For $t - \log(2) \leq s \leq t$ we have

$$\begin{aligned} d(\gamma_{o,\xi}(s), \gamma_{o,\eta}(s)) &\leq d(\gamma_{o,\xi}(s), \gamma_{o,\xi}(t - \log(2))) + \delta_0 + d(\gamma_{o,\eta}(s), \gamma_{o,\eta}(t - \log(2))) \\ &\leq \delta_0 + 2\log(2), \end{aligned}$$

concluding the claim.

Clearly, since \mathbb{H}^{n+1} is of constant sectional curvature, there exists a universal constant $k > 0$ such that the hyperbolic volume of $\mathcal{N}_{\delta_0 + 2\log(2)}(\gamma_{o,\eta}((t - c, t]))$ is bounded by $k \cdot c$. Since moreover \mathcal{C}_3 is τ_0 -separated it also follows that there exists a constant $\bar{k} = \bar{k}(\tau_0) > 0$ such that the (hyperbolic) volume of every ball $B(x, \tau_0/2)$ is at least \bar{k} . Thus, we conclude that $G(\eta, t, c) \leq k/\bar{k} \cdot c$, as stated above.

For the remaining part, since \mathcal{C}^3 is R_0 -spanning, consider an element $x \in \mathcal{C}^3$ such that $d(\gamma_{o,\eta}(t - R_0), x) \leq R_0$. Hence,

$$t - 2R_0 \leq d(o, x) \leq t.$$

Moreover, when t_3 is sufficiently large with respect to R_0 and a given $\varepsilon > 0$, then it follows from hyperbolic geometry that for some constant $\tilde{u}_* = \tilde{u}_*(\varepsilon, R_0)$ we have $d(\gamma_{o,x}(s), \gamma_{o,\eta}(s)) \leq \varepsilon$ for all $s \leq t - R_0 - \tilde{u}_*$. Thus, setting $u_* = R_0 + \tilde{u}_*$ for a suitable $\varepsilon > 0$, it follows from Lemma 2.1 that $\partial_\infty x = \gamma_{o,x}(\infty) \in B_{d_o}(\eta, e^{-(t-u_*)})$.

This finishes the proof. \square

2.4. Proof of Theorem 1.3 (Absolute winning). Let M be as in Section 1.2.1 which determines a collection \mathcal{C}^1 of pairwise disjoint horoballs as above. Identify $\partial C_o^1 \subset \mathbb{H}^{n+1}$ with \mathbb{R}^n via the map $x \mapsto \gamma_x(\infty) \in \mathbb{R}^n$ where γ_x is the vertical geodesic line in \mathbb{H}^{n+1} with $\gamma_x(0) = x \in \partial C_o^1$. Note that the cocompact torsion-free stabilizer $\Gamma_\infty = \text{Stab}_\Gamma(\infty)$ acts isometrically on \mathbb{R}^n and since C_o^1 is precisely invariant, the projection $\tilde{\pi}$ of a compact fundamental domain F of Γ_∞ in $\partial C_o^1 = \mathbb{R}^n$ locally embeds isometrically into $\partial H_0 \subset M$ (up to rescaling the length metric on ∂H_0). In particular, identify $\mathbb{R}^n/\Gamma_\infty$ with SH_0^+ via the map

$$F \ni x \mapsto d\tilde{\pi}(\dot{\gamma}_x(0)) \equiv v_x \in SH_0^+$$

which together with the translation to F determines the map in (1.6).

Let $c_0 > 0$ and recall that $\tilde{S}_{c_0} \subset \partial C_o^1 = \mathbb{R}^n$ denotes the lift of S_{c_0} , the set of vectors in SH_0^+ for which the first penetration height equals c_0 . From Lemma 2.1, $\gamma_x, x \in \mathbb{R}^n$, intersects a horoball C in \mathcal{C}^1 with height c_0 if and only if $\gamma_x(\infty)$ is contained in the $(n-1)$ -dimensional Euclidean sphere $S_C^{c_0} \equiv \partial B(\partial_\infty C, e^{-c_0}e^{-s_C}/2)$. Since $c_0 > 0$, if $\gamma_x(\infty) \in S_C^{c_0} \cap S_{\bar{C}}^{c_0}$ for $C, \bar{C} \in \mathcal{C}^1$ distinct and $s_C \leq s_{\bar{C}}$, then by the disjointness of C and \bar{C} the

ray γ_x hits C at a smaller time that it hits \bar{C} . Thus we see that \tilde{S}_{c_0} consists of a countable disjoint union of $(n - 1)$ -dimensional Euclidean spheres in \mathbb{R}^n .

Recall from [3] that a set $X \subset \mathbb{R}^n$ is b -diffuse for some $b > 0$ and $t_0 \in \mathbb{R}$ if, given any points $\bar{x} \in \mathbb{R}^n$ and $x \in X$, $t \geq t_0$, there exists a further point $y \in X$ such that

$$B(y, e^{-(t+b)}) \cap X \subset B(x, e^{-t}) \cap X - B(\bar{x}, e^{-(t+b)}).$$

We need the following.

Lemma 2.4. *Every $S = S_C^{c_0}$ is $\log(3)$ -diffuse for $t_C = s_C + c_0 + 2 \log(2)$.*

Proof. Consider two points $x \in S$ and $\bar{x} \in \mathbb{R}^n$. Given $t \geq t_C$, let $\bar{B} = B(\bar{x}, e^{-t}/3) \cap S$. Clearly, the worst case to consider is when \bar{x} actually lies on S . But for this case, since $e^{-t} \leq e^{-t_C} = r/4$ for the radius r of S , it is easy to see that there exists a point $y \in S$ such that $B(y, e^{-t}/3) \cap S \subset B(x, e^{-t}) \cap S - \bar{B}$, finishing the proof. \square

Recall from the author's earlier work [24], Theorem 3.11,⁸ that given the collection \mathcal{C}^1 as above and a b -diffuse set $X = S_C^{c_0} \subset \mathbb{R}^n$, the set $\tilde{S}(c_0) = \cup_{c>0} \mathbf{Bad}_X(\mathcal{R}^1, c)$ is absolute winning in X . This already finishes the proof.

Remark. Note that the first part of Proposition 2.3, Lemma 2.1 as well as [24], Theorem 3.11, also hold in curvature at most -1 . Moreover the above arguments translate in a similar way (if we replace spheres by submanifolds diffeomorphic to spheres) so that Theorem 1.3 holds for pinched negative curvature as well.

2.5. Proof of the Theorems 1.1, 1.5, 1.7 (Hall ray-type results). Given $C \in \mathcal{C}^i$ and $o \in \mathbb{H}^{n+1} \cup S^n$ as above, we define the following three maps $\mathfrak{p}_{C_o^i, C}^i : \bar{X}_i \rightarrow [0, \infty]$, where

1. $\mathfrak{p}_{C_o^1, C}^1(\xi) \equiv \mathcal{H}_{C_o^1, C}(\xi)$, where $\mathcal{H}_{C_o^1, C}(\xi) \equiv \sup_{t \in \mathbb{R}} \beta_C(\gamma_{C_o^1, \xi}(t))$,
2. $\mathfrak{p}_{C_o^2, C}^2(\xi) \equiv \mathcal{L}_{o, C}(\xi)$, where $\mathcal{L}_{o, C}(\xi) \equiv L(\mathcal{N}_{\varepsilon_0}(C) \cap \gamma_{o, \xi})$,
3. $\mathfrak{p}_{C_o^3, C}^3(\xi) \equiv \mathcal{D}_{o, C}(\xi)$, where $\mathcal{D}_{o, C}(\xi) \equiv -\log(d(C, \gamma_{o, \xi}))$.

Let $t_i \geq 0$ be technical constants, where $t_1 = t_2 = 0$ and $t_3 \geq \bar{c}_0$ for the constant \bar{c}_0 from Lemma 2.1. Choose one of the convex sets $C_0 = C_o^i \in \mathcal{C}^i$ with $s_0^i \equiv d(C_o^i, C_0)$ minimal under the condition $d(C_o^i, C_0) \geq t_i$ (which exists by discreteness). For the third Case assume $t_3 = s_0^3$ and note that t_3 corresponds to the constant t_0 in Section 1. Since we only consider geodesic rays starting from o for times $t \geq t_3$ and by the choices below, we may simply ignore all points $x \in \mathcal{C}^3$ with $d(o, x) < t_3$ in the following and delete them from the collection \mathcal{C}^3 . This follows from the next remark.

Remark. Let $x \in \mathcal{C}^3$ with $d(o, x) < t_3$, hence $x \neq C_0$. For any time $t \geq t_3$ and geodesic ray γ starting from o and with $d(\gamma, C_0) \leq e^{-c_0}$, we have

$$d(\gamma(t), x) \geq d(\gamma(t_3), x) \geq d(C_0, x) - d(\gamma(t_3), C_0) \geq \tau_0 - e^{-c_0} \geq \tau_0/2$$

for $c_0 \geq -\log(\tau_0/2)$.

For $c_0 > 0$ sufficiently large (with $c_0 \geq \bar{c}_0$), the main idea is to define the set

$$X_i \equiv (\mathfrak{p}_{o, C_0}^i)^{-1}(c_0),$$

in \bar{X}_i , which is diffeomorphic to a $(n - 1)$ -dimensional Euclidean sphere; see Lemma 2.7 below. By choice, for each of the cases, a ray $\gamma_{C_o^i, \xi}|_{[t_i, \infty)}$ with $\xi \in X_i$ will 'hit' first the set $C_0 \in \mathcal{C}^i$ and has exactly the desired penetration property with respect to the parameter c_0 .

⁸ The result in [24] is stated for a slightly different setup which however plays no role.

In view of Lemma 2.5 below, given a further large parameter s_0 , our aim is to show the existence of a subset A of X_i for which any given $\xi \in A$ satisfies

$$d_o(\xi, \partial_\infty C) \geq \kappa_u \cdot e^{-s_0} e^{-d(C_o^i, C)},$$

for all $C_0 \neq C \in \mathcal{C}^i$ where κ_u is from Lemma 2.1, and with a lower bound on the Hausdorff-dimension of A depending on the parameter s_0 .

More precisely, set $\bar{\kappa}_u \equiv -\log(\kappa_u)$ and choose $l_*^i \equiv l_i + \log(3)$, where l_i is given in Proposition 2.3 and $l_*^3 = \log(2)$. Given $s_0 \geq l_*^i - \bar{\kappa}_u$ with $s_0 \geq \bar{c}_0$ let $c \geq 0$ such that $s_0 + \bar{\kappa}_u = 2c + l_*^i$, that is

$$c = \frac{s_0 + \bar{\kappa}_u - l_*^i}{2}.$$

Then, we exclude the set C_0 from the collection \mathcal{C}^i and choose $A \equiv \mathbf{Bad}_{X_i}(\mathcal{F}^i, s_0 + \bar{\kappa}_u)$ and remark that, in fact, A projects (locally injectively) to a subset $S(c_0, s_0)$.

Lemma 2.5. *Given $\xi \in \mathbf{Bad}_{X_i}(\mathcal{F}, s_0 + \bar{\kappa}_u)$ we have $\mathbf{p}_{o, C_0}^1(\xi) = c_0$ and $\mathbf{p}_{o, C}^i(\xi) \leq s_0$ for all $C_0 \neq C \in \mathcal{C}^i$ with $d(C_o^i, C) \geq t_i$; in particular, $\gamma|_{[t_i, \infty)}$ projects to a geodesic in $S(c_0, s_0)$.*

Proof. By construction, every $\xi \in X_i$ satisfies $\mathbf{p}_{o, C_0}^i(\xi) = c_0$. Let $C_0 \neq C \in \mathcal{C}^i$ with $d(C_o^i, C) \geq t_i$. It follows from the definition of $\mathbf{Bad}_{X_i}(\mathcal{F}, s_0 + \bar{\kappa}_u)$ that $d_o(\xi, \partial_\infty C) \geq \kappa_u \cdot e^{-(s_C^i + s_0)}$. Thus, by Lemma 2.1, we have that

$$\mathbf{p}_{o, C}^i(\xi) \leq -\log\left(\frac{1}{\kappa_u} e^{s_C^i} d_o(\xi, \partial_\infty C)\right) \leq s_0,$$

as claimed. Recalling from the above remark that we may ignore all $x \in \mathcal{C}^3$ with $d(o, x) < t_3$, the proof follows. \square

Thus, a lower bound on the Hausdorff-dimension of A will be a lower bound on the dimension of $S(c_0, s_0)$. For the respective cases, set

$$t_*^i = t_*^i(c_0) \equiv s_0^i + c_0 + \log(\bar{c}_n) + \log(3) + \log(2), \quad \Omega_i \equiv X_i \times [t_*^i, \infty),$$

where $\bar{c}_n \geq 1$ is determined in the proof of Lemma 2.7 and independent of s_0^i, c_0 (and s_0).

In order to obtain a lower bound for $\dim(A)$, we check conditions (S0), ($\mu 1$) and ($\mu 2$). Recall that condition (S0) is trivially satisfied for all $(\xi, t_*^i) \in \Omega$ whenever $c \geq t_*^i - s_0^i$ (note again that $s_0^i = \min\{s_C^i : C \in \mathcal{C}^i\}$ for all three cases); hence for

$$s_0 \geq 2c_0 + 2|\log(6\bar{c}_n)| + l_*^i - \bar{\kappa}_u \equiv 2c_0 + k_1.$$

More generally, condition (S0) is satisfied in the following situations.

Lemma 2.6. *For Cases 1, 2, when c is sufficiently large (independent on c_0), that is when*

$$e^{s_0} e^{-l_i} (1 - e^{-c} \cdot \kappa_u e^{\log(6\bar{c}_n)}) \geq \kappa_u,$$

then for all $C_0 \neq C \in \mathcal{C}^i$ with $s_C^i \leq t_ - c - l_*^i$ we have*

$$d_o(X_i, \partial_\infty C) \geq \kappa_u \cdot e^{-(s_C^i + s_0)};$$

in particular (S0) is satisfied for any $(\xi, t_^i) \in \Omega_i$.*

Proof. Let $\xi \in X_i$ be any point. Given $C_0 \neq C \in C^i$ with $s_0^i \leq s_C^i \leq t_* - l_*^i - c$, using Proposition 2.3 and Lemma 2.1, we have

$$\begin{aligned} d_o(\xi, \partial_\infty C) &\geq d_o(\partial_\infty C_0, \partial_\infty C) - d_o(\partial_\infty C_0, \xi) \\ &\geq e^{-l_i} e^{-s_C^i} - \kappa_u e^{-(s_0^i + c_0)} \\ &\geq e^{-(s_C^i + s_0)} \cdot e^{s_0} (e^{-l_i} - \kappa_u e^{s_C^i - (s_0^i + c_0)}) \\ &\geq e^{-(s_C^i + s_0)} \cdot e^{s_0} (e^{-l_i} - \kappa_u e^{t_* - (s_0^i + c_0 + c + l_*^i)}) \\ &= e^{-(s_C^i + s_0)} \cdot e^{s_0} e^{-l_i} (1 - e^{-c} \cdot \kappa_u e^{\log(\bar{c}_n) + \log(6)}) \equiv e^{-(s_C^i + s_0)} \cdot h_*, \end{aligned}$$

where $h_* = h_*(s_0) \geq \kappa_u$ is independent on c_0 . \square

We need to establish the following crucial result.

Lemma 2.7. *X_1, X_3 are isometric to and X_2 is diffeomorphic to a $(n-1)$ -dimensional Euclidean sphere. Moreover, there exist measures μ_i such that $(\Omega_i, \psi_i, \mu_i)$ satisfies a power law with respect to the exponent $\tau = n-1$ and constants $c_u = \bar{c}_n \cdot e^{-(n-1)(s_0^i + c_0)}$, $c_l = \bar{c}_n^{-1} \cdot e^{-(n-1)(s_0 + s_1)}$ where $\bar{c}_n \geq 1$ is independent from s_0^i and c_0 ; hence (μ_1) is satisfied.*

Proof. For the second case assume that $\partial_\infty C_0^2$ equals $\{0, \infty\}$ and let x_2 denote the (unique) point on the vertical line C_0^2 at distance $d(o, C_0^2) = s_0^2$ to o . For the third case assume $o = e_{n+1}$ and $\partial_\infty C_0^3 = 0 \in \mathbb{R}^n$. We may also assume that $x_2 = e_{n+1}$ and in addition, for c_0 sufficiently large, that X_2 and X_3 are contained in the unit ball around $0 \in \mathbb{R}^n$ on which $d_{e_{n+1}}$ is c_B -bi-Lipschitz equivalent to the Euclidean metric for some $c_B \geq 1$, see (2.1).

From Lemma 2.1 we know that $X_1 = \partial B(\partial_\infty C_0^1, e^{-c_0} e^{-s_1}/2)$ is a $(n-1)$ -dimensional Euclidean sphere. For the third case, it follows from symmetry that $X_3 = \partial B(0, r_3)$ is as well a $(n-1)$ -dimensional Euclidean sphere. For the second case, denote for a point $x \in \mathbb{H}^{n+1}$ the set of $\xi \in \mathbb{R}^n$ for which the penetration length of $\gamma_{x,\xi}$ in $\mathcal{N}_{\varepsilon_0}(C_0)$ equals c_0 by $S_x(c_0) = (\mathfrak{p}_{x,C_0}^2)^{-1}(c_0) \subset \mathbb{R}^n$. Since $x_2 \in C_0$, $S_{x_2}(c_0)$ is again by symmetry a $(n-1)$ -dimensional Euclidean sphere $\partial B(\eta_0, r_2)$. Moreover, for c_0 sufficiently large, $S_x(c_0)$ is a submanifold which varies smoothly in x , showing that $X_2 = S_o(c_0)$ is diffeomorphic to the Euclidean sphere $S_{x_2}(c_0)$. Note that the visual metrics d_0 and $e^{-d(o,x_2)} \cdot d_{x_2}$ are bi-Lipschitz equivalent (with a constant independent on $d(o, x_2) = s_0^2$). It follows from Lemma 2.1 that $(X_i, d_o|_{X_i \times X_i})$ is L -bi-Lipschitz homeomorphic to a Euclidean sphere $\partial B(0, r_i)$ with the induced Euclidean metric and of radius $r_i = e^{-(s_0^i + c_0)}$, where $L \geq 1$ is independent of s_0^i and c_0 ; let $f_i : \partial B(0, r_i) \rightarrow X_i$ denote this homeomorphism.

Define $S_r^{n-1} \equiv \partial B(0, r) \subset \mathbb{R}^n$ an Euclidean sphere of radius r . For the unit sphere S_1^{n-1} with the angle metric the Lebesgue measure μ , restricted to balls of radius at most $\pi/8$, clearly satisfies a power law with exponent $n-1$; that is, $c_l R^{n-1} \leq \mu(B(x, R)) \leq c_u R^{n-1}$ for multiplicative constants $c_u \geq c_l > 0$ and all balls $B(x, R) \subset S_1^{n-1}$ with $R \leq \pi/16$. For $S_r^{n-1} \subset \mathbb{R}^n$ with the induced metric the (radial) projection map $f_r : S_1^{n-1} \rightarrow S_r^{n-1}$ is a $2r$ -bi-Lipschitz homeomorphism, restricted to balls of radii at most $\pi/16$ and $r/16$ respectively. Thus, the push-forward measure $(f_r)_* \mu$ supported on S_r^{n-1} , restricted to balls of radius at most $r/32$, satisfies also a power law with exponent $n-1$ and multiplicative constants $c_u = \bar{c}_u r^{n-1}$, $c_l = \bar{c}_l r^{n-1}$ where \bar{c}_u, \bar{c}_l are independent of r .

Finally, it is readily checked that the push forward measures $\mu_i \equiv (f_i \circ f_{r_i})_* \mu$ on $(X_i, d_o|_{X_i \times X_i})$ give the desired measures, restricted to balls of radius at most r_i/\bar{c}_n where $\bar{c}_n \geq 1$ is sufficiently large, depending only on \bar{c}_u, \bar{c}_l, L and c_B . \square

Finally, we determine the following parameters for (μ_2) .

Lemma 2.8. For $c \geq d_* + d_c$, $(\Omega_i, \psi_i, \mu_i)$ is τ_c -decaying with respect to \mathcal{R}^i where

$$\tau_c^1 = \tau_c^2 \equiv \bar{c}_n^2 e^{(n-1)(d_*+d_c)} \cdot e^{-(n-1)c}, \quad \tau_c^3 = \bar{c}_n^2 k_0 e^{(n-1)(d_*+d_c)} \cdot (c + l_*^3) \cdot e^{-(n-1)c},$$

and k_0 denotes the constant from Proposition 2.3.

Proof. Let $\omega = (\xi, t + d_c) \in \Omega_i$. For the first and second case, we know from Proposition 2.3 that distinct $\partial_\infty C, \partial_\infty \bar{C}$ in $\mathcal{R}^i(t - l_*^i)$ satisfy

$$d_o(\partial_\infty C, \partial_\infty \bar{C}) \geq e^{-l_i} e^{-\max\{s_C^i, s_{\bar{C}}^i\}} \geq 3 \cdot e^{-t},$$

since $s_C^i, s_{\bar{C}}^i \leq t - l_*^i \leq t - l_i - \log(3)$. Hence, at most one point of $\mathcal{R}^i(t - l_*^i)$ can lie in the ball $B_{d_o}(\xi, 1.5e^{-t})$. In particular, for $c \geq d_* = \log(3)$, for at most one such point η of $\partial_\infty C, \partial_\infty \bar{C}$, the ball $B_{d_o}(\eta, e^{-(t+c-d_*)}) \subset B_{d_o}(\eta, e^{-t}/3)$ can intersect $B_{d_o}(\xi, e^{-t}) \supset \psi(\omega)$. Let η be such a point and note that the measure of $B_{d_o}(\eta, e^{-(t+c-d_*)}) \cap X_i$ is clearly maximized when $\eta \in X_i$. Thus, since $(\Omega_i, \psi_i, \mu_i)$ satisfies a power law, we have

$$\begin{aligned} \mu(\psi(\omega) \cap \mathcal{N}_{e^{-(t+c-d_*)}}(\mathcal{R}^i(t - l_*^i))) &\leq \mu(X_i \cap B(\eta, e^{-(t+c-d_*)})) \\ &\leq c_u e^{-(n-1)(t+c-d_*)} \\ &\leq \frac{c_u}{c_l} e^{-(n-1)(c-d_*-d_c)} \cdot c_l e^{-(n-1)(t+d_c)} \leq \tau_c^i \cdot \mu(\psi(\omega)), \end{aligned}$$

showing the claim.

For the third case, consider the ball $B = B_{d_o}(\xi, 2e^{-t}) = B_{d_o}(\xi, e^{-(t-l_*^3)}) \supset \psi(\omega)$. From Proposition 2.3 we know for all $c \geq 0$ that

$$|\{\partial_\infty x \in B : x \in \mathcal{C}^3 \text{ with } d(o, x) \in [t - c, t]\}| \leq k_0 \cdot c.$$

Then, with the same arguments as above, we have

$$\begin{aligned} \mu(\psi(\omega) \cap \mathcal{N}_{e^{-(t+c-d_*)}}(\mathcal{R}^3(t - l_*^3, c))) &\leq \sum_{\partial_\infty x \in B: d(o, x) \in [t-l_*^3-c, t-l_*^3]} \mu(X_3 \cap B_{d_o}(\xi_\infty, e^{-(t+c-d_*)})) \\ &\leq c_u k_0 \cdot (c + l_*^3) \cdot e^{-(n-1)(t+c-d_*)} \\ &\leq \frac{c_u}{c_l} k_0 \cdot (c + l_*^3) \cdot e^{-(n-1)(c-d_*-d_c)} \cdot c_l e^{-(n-1)(t+d_c)} \\ &\leq \tau_c^3 \cdot \mu(\psi(\omega)), \end{aligned}$$

finishing the proof. \square

Assume that s_0 (and hence $c = (s_0 + \bar{\kappa}_u - l_*^i)/2$) is sufficiently large as above and independently from c_0 such that $\tau_c^i < 1$ as well as

$$|\log(1 - \tau_c^i)| \leq \frac{1}{4} \log(2\bar{c}_n^4 e^{(n-1)(d_*+d_c)}) \leq \frac{1}{4} \log(\bar{c}_n^4 2^n 3^{n-1}).$$

Summarizing, when both $c_0, s_0 \geq \bar{t}_0$ are sufficiently large, Proposition 2.2 implies that for all three cases

$$\begin{aligned} \dim(\mathbf{Bad}_{X_i}(\mathcal{R}^i, 2c + l_*^i)) &\geq \tau - \frac{\log(2\bar{c}_n^4 e^{\tau(d_*+d_c)}) + |\log(1 - \tau_c^i)|}{c} \\ &\geq (n-1) - \frac{\log(\bar{c}_n^4 2^n 3^{n-1})}{2(\frac{s_0 + \bar{\kappa}_u - l_*^i}{2})} \\ &\geq (n-1) - \frac{\bar{k}_0}{s_0}, \end{aligned}$$

for a suitable constant $\bar{k}_0 = \bar{k}_0(\bar{t}_0) > 0$ independent of c_0, s_0 . This finishes the proofs.

Remark. For $n = 2$, X_1 is a Euclidean sphere S^1 of radius $e^{-(s_1+c_0)}/2$ in which balls can be subpartitioned nicely. Following again [26], Section 2.2.1, the lower bound can be improved to $1 - \frac{\bar{k}_0}{s_0 \cdot e^{s_0}}$ for some $\bar{k}_0 > 0$.

2.6. Proof of Theorem 1.4 (Jarník-type inequality). Assuming that we are given the collection \mathcal{C}^3 we let $t_* = t_0 \geq \bar{c}$ (as in Lemma 2.1 and Proposition 2.3) be sufficiently large and $X_3 = S^n$ be the full boundary. Since we only consider subrays $\gamma_v|_{[t_0, \infty)}$, it is readily checked that points $x \in \mathcal{C}^3$ with $d(o, x) \leq t_* - 1$ will play no role and we may hence exclude them from the collection \mathcal{C}^3 . Let μ be the Lebesgue measure on S^n for which (Ω, ψ, μ) satisfies a power law with respect to the exponent n and positive constants c_l, c_u . We need the following Lemma.

Lemma 2.9. *For $c > d_*$, (Ω, ψ, μ) is τ_c -decaying with respect to \mathcal{R} for*

$$\tau_c = \frac{c_u}{c_l} k_0 e^{n(d_*+d_c)} \cdot (c + l_*) \cdot e^{-nc}, \quad (2.7)$$

where $l_* = l_*^3$ is as above.

Moreover, given $B = B_{d_o}(\eta, e^{-(t-u_*-d^c)})$ where $(\eta, t - u_* - d^c) \in \Omega_3$, we have

$$\mu(B \cap \bigcup_{\partial_\infty x \in \mathcal{R}(t)} B(\partial_\infty x, e^{-(s_x+c+d_*)})) \geq \bar{k}_u e^{-nc} \cdot \mu(B) \equiv \tau^c \cdot \mu(B), \quad (2.8)$$

where \bar{k}_u denotes a constant independent on $c > 0$, u_* is the constant from Proposition 2.3 and $d^c = \log(1 - e^{-c})$.

In the language of [26], (2.8) means that (Ω, ψ, μ) is τ^c -Dirichlet with respect to \mathcal{R}^3 and the parameters (c, u^*) . Moreover, all the requirements are satisfied to apply the abstract formalism of [26] for the upper bound to our setting.

Proof of Lemma 2.9. The first part follows in a similar way to the proof of Lemma 2.8.

For the second part, given B from the statement, Proposition 2.3 shows that there exists a point $x \in \mathcal{C}^3$ such that $\partial_\infty x \in B_{d_o}(\eta, e^{-(t-u_*)})$ with $t - u_* \leq s_x \leq t$. Hence, by definition of d^c , we have $B_{d_o}(\partial_\infty x, e^{-(s_x+c)}) \subset B_{d_o}(\partial_\infty x, e^{-(t-u_*+c)}) \subset B$. Thus, we see

$$\begin{aligned} \mu(B \cap \bigcup_{\partial_\infty y \in \mathcal{R}(t)} B(\partial_\infty y, e^{-(s_y+c+d_*)})) &\geq \mu(B_{d_o}(\partial_\infty x, e^{-(s_x+c+d_*)})) \\ &\geq c_l e^{-n(s_x+c+d_*)} \\ &\geq \frac{c_l}{c_u} e^{n((t-s_x)-(c+u_*+d^c+d_*))} \cdot c_u e^{-n(t-u_*-d^c)} \\ &\geq \frac{c_l}{c_u} e^{-n(c+u_*+d^c+d_*)} \cdot c_u e^{-n(t-u_*-d^c)} \geq \tau^c \cdot \mu(B), \end{aligned}$$

as claimed. \square

Clearly, since $t_* = t_0$ and $\min\{s_x : x \in \mathcal{C}^3\} \geq t_* - 1$, (S0) is trivially satisfied for all $\omega_1 = (\xi, t_*) \in \Omega$ when restricting to $c \geq 1$. We may assume that $\xi = 0 \in \mathbb{R}^n$ and let $c_B \geq 1$ be the constant such that d_o , restricted to the unit ball in \mathbb{R}^n , is bi-Lipschitz equivalent to the Euclidean metric d_E . Since t_* is sufficiently large, we may assume that $B_{d_o}(0, 2e^{-t_*}) \subset B(0, 1)$ is contained in the Euclidean unit ball.

In particular, (2.4) and (2.8) also hold for the Euclidean metric and the Lebesgue measure on \mathbb{R}^n , up to a multiplicative constant depending on c_B . Finally, we remark that (2.4) and (2.8) also remain true, up to a further multiplicative constant (that is with respect to $\bar{\tau}_c = k_l \cdot c \cdot e^{-nc}$ and $\bar{\tau}^c = k_u e^{-nc}$ respectively), if we replace the Euclidean balls $B(x, r)$ by Euclidean cubes $Q(x, r) = [x_1 - r, x_1 + r] \times \cdots \times [x_n - r, x_n + r]$, centered at $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with the same radius $r > 0$; see the Lemmata 2.5 and 2.6 in [26].

Thus, for $N_0 \leq m \in \mathbb{N}$ with N_0 sufficiently large we let $c = \log(m)$ or $c + u^c = \log(m)$ for some u^c such that $u^* \leq u^c \leq u^* + \log(2)$ and can in fact apply the standard cases of [26], Sections 2.2.1 and 2.3.1. From these we obtain

$$\begin{aligned} \dim(\mathbf{Bad}_{S^n}(\mathcal{R}, 2c + l_*^3)) &\geq n - \frac{|\log(1 - \bar{\tau}_c)|}{c} \\ &= n - \frac{|\log(1 - k_l \cdot c \cdot e^{-nc})|}{c} \end{aligned}$$

as well as

$$\begin{aligned} \dim(\mathbf{Bad}_{S^n}(\mathcal{R}, c) \cap B_{d_o}(\xi, e^{-t_*})) &\leq n - \frac{|\log(1 - \bar{\tau}^c)|}{c + u^c} \\ &\leq n - \frac{|\log(1 - k_u \cdot e^{-nc})|}{c + u^* + \log(2)}. \end{aligned}$$

Since the argument is independent from the chosen formal ball $\omega_1 = (\xi, t_*)$ and using the countable stability of the Hausdorff-dimension, the upper bound holds for $\mathbf{Bad}_{S^n}(\mathcal{R}, c)$. Finally, applying the Taylor expansion and using Lemma 2.1 finishes the proof.

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